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INEQUALITIES FOR ORTHOGONAL POLYNOMIALS AND BESSEL FUNCTION

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***Abstract*—** In this paper, we have given several proofs inequality discovered by Turban for Legendre polynomials. A single derivation and some their results were given. We also established in equality for bassel function. We illustrate and rederive the left hand inequality further more to establish the estimate for 0(x).

***Keywords*— Laguerre and Hermite polynomials, Bessel function, Turan's inequality, ultra spherical polynomials.**

I. INTRODUCTION

In a paper published some time ago, szegö [I] has given several proofs of the following – interesting inequality discovered by Turan for Legendre polynomials $P_n(x)$:

(1.1) $\Delta_n(x) \equiv P_n^2(x) - P_{n-1}(x) P_{n+1}(x) \geq 0, n \geq 1, -1 \leq x \leq +1.$

Szegö remarks further that inequalities analogous to (1.1) hold also for the ultraspherical, Laguerre and Hermite polynomials. Thus for the ultraspherical polynomials $P_{n,\lambda}$ (x), the analogue of (1.1) reads:

(1.2) $\Delta_{n,\lambda}(x) \equiv P_{n,\lambda}^2(x) - F_{n-1,\lambda}(x) F_{n+1,\lambda}(x) \geq 0, \lambda > -\frac{1}{2}, n \geq 1, |x| \leq 1.$

Where $F_{n,\lambda}(x) \equiv P_{n,\lambda}(x)/P_{n,\lambda}(1)$. Subsequently, Madhava Rao and Thiruvenkatachar [2] showed that an elementary proof of (1.1) may be obtained by merely finding $\Delta_n^{''}(x)$, which is given by the elegant formula

(1.3) $\Delta_n^{''}(x) = -\frac{2}{n(n-1)} \{P_n^{'}(x)\}^2$

Similar proofs are also set forth in [2] for the Laguerre and Hermite cases. A simple derivation of (1.3) and some other results were given later by one of us [3]. Recently, Szász [4] has derived a sharper inequality than (1.2) for o < λ < 1 :

(1.4) $\frac{\lambda(10F_{n,\lambda}^2(x))}{(n+\lambda-1)(n+2\lambda)} \leq \Delta_{n,\lambda}(x) < \frac{n+\lambda}{\lambda+1} \; \frac{\Gamma(n)\Gamma(2\lambda+1)}{\Gamma(n+2\lambda+1)}$

Which, in the Legendre case (λ = ½) reads:

(1.5) $\frac{1-P_{n,}^2(x))}{(2n-1)(n+1)} \leq \Delta_{n,}(x) < \frac{2n+1}{3n(n+1)}$

He has also established the following inequality for the Bassel function $J_r(x)$:

(1.6) $J_r^2(x) - J_{v-1}(x) J_{v+1}(x) \geq \frac{1}{v+1} \; J_r^2(x), v \geq 0, -\infty < x < +\infty.$

The procedure adopted by Szász in deriving (1.4) and (1.6) is based mainly on the respective recurrence relations satisfied by $P_{n,\lambda}(x)$ and $J_v(x)$. It has the merit of naturally leading to the more refined inequalities obtained by him, but the results themselves, when known, should be capable of a shorter and more direct proof. We illustrate this in this note by utilising the results of [2] and [3]

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to rederive the left hand inequality in (1.5) and furthermore, to establish the following estimate for $\Delta_n(x)$ which is at once simpler and more precise than (1.5) :

(1.7) $\frac{1-x^2}{2n} \le \Delta_n(x) < \frac{2}{\pi n};$

Here the constant 2/π cannot be replaced by a smaller one. We also set out certain other inequalities which are of interest in this context. To this end, we consider the function. By the same differentiation method, we establish the inequality:

(1.8) $J_\gamma^2(x) - J_{\gamma-1}(x) J_{\gamma+1}(x) \ge 0.$ $\gamma \ge -1$

From which we readily deduce (1.6) by using the recursion for $J_\gamma(x)$. On the other hand, for the modified Bessel function $I_\gamma(x)$ we prove the inequality:

(1.9) $O \le I_\gamma^2(x) - I_{\gamma-1}(x) I_{\gamma+1}(x) \le \frac{1}{\gamma+1} I_\gamma^2(x), \gamma > -1.$

Finally, we derive some identities where by the inequality (1.2), the analogous inequality for the Laguerre polynomials $L_n^{(\alpha)}(x)$ and lastly, the inequality (1.6) for Bessel functions are all rendered intuitive. These identities deduced merely by means of the respective recurrence relations were suggested by a known identity for Hermite polynomials mentioned by Szasz in his paper ([4], P. 264). Herewith we secure perhaps the simplest proofs of the inequalities in question.

2. LEGENDRE POLYNOMIALS

We begin with the left hand inequality in (1.5). In fact, we establish a more complete inequality, viz.,

(2.1) $\Delta_n(x) \lesseqgtr (x) \frac{1-p_n^2(x)}{(2n-1)(n+1)}$ according as $|x| \lesseqgtr 1$, (n>1).

(2.2) $f_n(x) = \Delta_{n,}(x) - \frac{1-P_{n,}^2(x)}{(2n-1)(n+1)}$ $n>1.$

Differentiating this twice and using (1.3), we obtain on simplification

$$f_n''(x) = \frac{2}{n(n+1)(2n-1)} \left\{ nP_n P_n'' - (n-1) P_n''^1 \right\}$$

Now we know that the roots of $P_n(x)$ are all real and simple. Denoting them by $x_{\gamma n}$ (γ=1,. . ., n) ,

we have $\frac{P_n'}{P_n} = \sum_{v=1}^{n} \frac{1}{x - x_{vn}}, \frac{P_n'^2 - P_n P_n''}{P_n^2} = \sum_{v=1}^{n} \frac{1}{(x - x_{vn})^2}$

Giving

(2.3) $f_n''(x) = \frac{2}{n(n+1)(2n-1)} \left\{ \left(\sum_{v=1}^{n} P_{vn} \right)^2 - n \sum_{v=1}^{n} P_{vn}^2 \right\}$

Where the $P_{\gamma n}$ are real polynomials defined by

$P_n(x) = (x - x_{\gamma n}) P_{vn}(x)$, ($\gamma$=1,…,n)

Now by Cauchy’s inequality (2.3) gives f_n''(x) < 0, so that f_n'' (x) is decreasing for all x. Since $f_n(x)$ is an event function, f_n' (x) is an odd function so that f_n' (0) = 0 It follows that f_n'' (x) < O according as x ≤ 0, so that $f_n(x)$ is increasing x < o, decreasing x > o and has its maximum for x o. Since $f_n(-1) = f_n(+1) = 0$, (2.1) follows.

Turning to the right hand inequality in (1.7), we first recall the result which has been obtained in [3] by arguing with $\Delta_n(x)$ on the basis of (1.3) as we have done above with $f_n(x)$ on the basis of (2.3), that $\Delta_n(x)$ has its maximum for x = 0. $\Delta_n(x) \le \Delta_n(0)$ for all x.

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Now we have

$n\Delta_n(0) = 2m\ g_m^2$ (n = 2m-1 or 2m), with $g_m = \dfrac{1.3.5,....(2m-1)}{2.4.6...2m}$

Since it is easily seen that m $g_m^2 \uparrow 1/\pi$ as m $\to \infty$, we find that n $\Delta_n(0) \uparrow 2/\pi$ as n$\to \infty$. This establishes the inequality in question as well as the fact that $2/\pi$ is the best possible constant thereof. To prove the left hand inequality in (1.7), we make use of the result derived in [3] that the function

(2.4) $D_n(x) \equiv \{P_n'(x)\}^2 - P_{n-1}'(x)P_{n+1}'(x)$

has its minimum for x = 0 and is connected with $\Delta_n(x)$ by means of the relation

(2.5) $n(n+1)\Delta_n(x) = (1-x^2)D_n(x)$

Herewith we obtain the inequality:

(2.6) $\Delta_n(x) \begin{smallmatrix}\ge\\ -\\ <\end{smallmatrix} (1-x^2)\Delta_n(0)$ according as $|x| \begin{smallmatrix}\ge\\ -\\ <\end{smallmatrix} 1$. (n>1). Since we have already seen that $n\Delta_n(0)$ is a nondecreasing function of n we have $n\Delta_n(0) \ge \Delta_1(0)\ \dfrac{1}{2}$. We now get the inequality in question for $-1 \le x \le +1$. Note that for $-1 < x < +1$, the lower bound for $\Delta_n(x)$ furnished by (1.7) is better than that given by (1.5), the former being 0(1/n) while the latter is $O(1/n^2)$ since, for $|x| < 1$, $P_n^2 \to 0$ as $n \to \infty$.

3. BESSEL FUNCTIONS

The Bessel function $J_\gamma(x)$ of order γ $(-\infty < \gamma < +\infty)$ is defined by

(3.1) $J_\gamma(x) = \sum_{n=o}^{\infty}(-1)^n \dfrac{C_{r,n}}{n!}\left(\dfrac{x}{2}\right)^{\gamma+2n}$, $\quad c_{\gamma,n} = \dfrac{1}{\Gamma(\gamma+n+1)}$

With the understanding that for $\gamma \le -1$ we have written

$$C_{\gamma,n} = \frac{(\gamma+n+1)....(\gamma+[-\gamma])}{\Gamma(\gamma+[-\gamma]+1)}, \quad 0 \le n < [-\gamma]$$

The modified Bessel function $I_\gamma(x)$ is then given by

(3.2) $I_\gamma(x) = \sum_{n=o}^{\infty} \dfrac{C_{r,n}}{n!}\left(\dfrac{x}{2}\right)^{\gamma+2n}$,

If we write $J_\gamma(x) = x_\gamma g_\gamma(x)$, then $g_\gamma(x)$ is an even entire function. It is essential to interpret, in what follows $J_{\gamma-k}\ J_{\gamma+k}$ as $(x^2)^\gamma\ g_{\gamma-k}\ g_{\gamma+k}$ with a similar interpretation for $I_{\gamma-k}\ I_{\gamma+k}$. With this understanding consider the function

(3.3) $Q_v(x) = x^2(J_\gamma^2 - J_{\gamma-1}J_{\gamma+1}) \equiv (x^2)^{\gamma+1}(g_\gamma^2 - g_{\gamma-1}g_{\gamma+1}), \gamma \ge -1.$

Which is continuous for all x. Substituting for $J_{\gamma-k}\ J_{\gamma+k}$ in terms of J_γ, J_γ' we have

$$Q_\gamma(x) = x^2J_\gamma^2 - (\gamma J_\gamma + xJ_\gamma')(\gamma J_\gamma - xJ_\gamma')$$

$$= (x^2 - \gamma^2)\ J_\gamma^2 + x^2 J_\gamma'^2$$

Now $\phi_\gamma^1(x) = 2xJ_\gamma^2 + 2J_\gamma^1\ \{x^2J_\gamma'' + xJ_\gamma^1 + (x^2-\gamma^2)J_\gamma\}$

$$= 2xJ_\gamma^2 \equiv x^2(x^2)^\gamma g_\gamma'^2$$

This shows that $\phi_\gamma(x)$ is increasing for x > 0 and decreasing for x > 0 Since $\phi_\gamma(=) = 0$, we get $\phi_\gamma(x) \ge 0$, $(-\infty < x < +\infty)$ with equality only for x = 0. This proves (1.8).

Now the recursion formula for the J_γ gives at once

(3.4) $\gamma J_\gamma(J_\gamma + J_{\gamma+2}) = (\gamma+1)J_{\gamma+1}(J_{\gamma-1} + J_{\gamma+1})$

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Hence, on using the inequality (1.8) just proved, we get

$$\gamma J_{\nu}^{2} - (v+1) J_{v-1} J_{v+1} = J_{\nu+1}^{2} + v (J_{\nu+1}^{2} - J_{v} J_{v+2}) \geq 0, \nu \geq 0.$$

With equality only for x = 0. Herewith, Szász's inequality (1.6) is established.
To prove the analogous inequality (1.9) for $I_v(x)$, we use the formula

(3.5) $I_\lambda(x)\, I_\mu(x) = \sum_{n=o}^{\infty} \binom{\lambda+\mu+2n}{n} C_{\lambda,n} C_{\mu,n} \left(\frac{x}{2}\right)^{\lambda+\mu+2n}$, ,

Which one obtains by forming the Cauchy product of the series for I_λ and I_μ. The inequality to be proved now follows from (3.6) : $0 < C_{\nu,n}^{2} - C_{r-1,n}\, C_{r+1,n} \leq \frac{1}{r+1} C_{\nu,n}^{2} = v > -1, n \geq 0$. Which simply amounts to

$$0 < 1 - \frac{\nu+n}{\nu+n+1} \leq \frac{1}{r+1}, \quad v > 1, n \geq 0$$

The same argument actually proves the following more general inequality:

(3.7) $I_{\gamma-R+1}(x) I_{\gamma+R-1}(x) - I_{\gamma-R}(x) I_{\gamma+k}(x) \leq \frac{2k-1}{\gamma+k} I_{\gamma+k+1}(x) I_{\gamma+k+1}(x) I_{\gamma+k-1}(x), \; 1 \leq k < \nu+2$,

In particular when γ is a positive integer n, (3.7) easily gives

(3.8) $I_o\, I_{2n} \leq I_{n}^{2} \leq \binom{2n}{n} I_o\, I_{2n}$

4. REMARKS ON TURÁN's INEQUALITY FOR $p_n(x)$ and $L_n(x)$ =

We return for a moment to Turan's inequality (1.1) for the Legendre polynomials $P_n(x)$. For |x|>| the inequality is reversed as has been pointed out in [3]. :

(4.1) $P_{n}^{2}(x) - P_{n-1}(x) P_{n+1}(x) < 0, n \geq 1. \; |x| > 1.$

In this case, we may prove a generalisation of (4.1) analogous to the left hand inequality in (3.7) viz.,

(4.2) $P_{n-k+1}(x) P_{n+k-1}(x) - P_{n-k}(x) P_{n+k}(x) < 0, 1 \leq k \leq n, |x| > 1.$

We first observe that

(4.3) $P_{n-\gamma}(x) P_{n+\gamma}(x) > 0, 1 \leq \gamma \leq n, |x| > 1$

To prove (4.2), we use an induction on k and assume (4.2) to hold for some k (1 ≤ k ≤ n). From (4.1) for n – k and n+k respectively, we have, for |x| > 1,

$$P_{n-k}^{2} \; P_{n+k}^{2} < P_{n-k-1} P_{n-k+1} \; P_{n+k-1} \; P_{n+k+1}$$
$$< P_{n-k-1} P_{n-k} \; P_{n+k} \; P_{n+k+1}$$

By (4.3) and the induction hypothesis. Hence, again by (4.3)

$$P_{n-k} P_{n+k} < P_{n-k-1} \; P_{n+k+1}$$

Which is (4.2) for k+1. Since (4.2) is (4.1) for k = 1, the proof of (4.2) is complete. For a different proof of (4.2), see [3].
For the Laguerre polynomials $L_n(x)$, the analogue of (1.1) reads:

(4.4) $L_{n}^{2}(x) - L_{n-1}(x) L_{n+1}(x) \geq 0, n \geq 1, -\infty < x < +\infty$

This has been established in [2] for x ≥ 0. We shall now prove that it holds also for $x < 0$.
From the explicit representation of $L_n(x)$ viz.,

(4.5) $L_n(x) = \sum_{\gamma=o}^{n} \binom{n}{\gamma} \frac{(-x)^{\nu}}{\gamma!}$

It is clear that $L_n(x)$ is an increasing function of n for $x < 0$; we have, in fact,

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$L_n(x) - L_{n-1}(l) = \sum_{\gamma=o}^{n} \binom{n-1}{\gamma-1} \frac{(-x)^{\nu}}{\gamma!} > 0$ for x < 0.

Hence the function $l_n(x) = e^{-\frac{x}{2}} L_n(x)$ introduced in [2] is also an increasing function of n for $x < 0$.

Now for the function

(4.6) $f(x) = l_n^2 - l_{n-1}\, l_{n+1} = e^{-x}(L_n^2 - L_{n-1} L_{n+1})$

The following result was derived in [2] :

(4.7) $xf'(x) = (l_{n+1} - l_n)(l_n - l_{n-1})$

We therefore have $f'(x) < 0$ for $x < 0$. So that $f(x)$ is – decreasing for $x < 0$. Since f(o) = 0 it follows that $f(x) > 0$ for $x < 0$. This completes the proof of (4.4).

Since $L_n(x) > 0$ for $x < 0$ as is obvious from (4.5).

We have $L_{n-k}(x)\, L_{n+k}(x) > 0$ for $x < 0$.

Hence, by the inductive argument used above, for proving (4.2), we get

(4.8) $L_{n-k+1}(x)\ L_{n+k-1}(x) - L_{n-k}(x)\, L_{n+k}(x) > 0,\ 1 \le k \le n, x < 0$

5. ULTRASPHERICAL POLYNOMIALS

We have generalise part of the results obtained in [3] for the Legendre polynomials to derive upper and lower bounded for $\Delta_{n,\lambda}(x)$ in the case of ultraspherical polynomials. The results will be found to be analogous to the inequality (2.6).

The ultraspherical polynomials $P_{n,\lambda}(x)$ satisfy the relations

(5.1) $(1-x^2)\, P'_{n,\lambda} = (n+2\lambda-1)\, P_{n-1,\lambda} - nx\, P_{n,\lambda} = (n+2\lambda)\, x\, P_{n,\lambda} - (n+1)\, P_{n+1,\lambda}$ and the differential equation

(5.2) $(1-x^2)P'_{n,\lambda}\ \ (2\lambda+1)\, x\, P'_{n,\lambda} + n\,(n+2\lambda)\, P_{n,\lambda} = 0$

If we now recall the definition of $\Delta_{n\lambda}(x)$, and introduce the numbers

(5.3) $k_{n,\lambda} = n\,(n+2\lambda)\, P^2_{n,\lambda}\ (1) = (n+2\lambda-1)\,(n+1)\, P_{n-1\lambda},\ (1)\, P_{n+1,\lambda}\ (1)$ we have

(5.4) $k_n,\ \Delta_{n\lambda}(x) \equiv n\,(n+2\lambda)\, P^2_{n,\lambda}\ (x) - (n+2\lambda-1)\,(n+1)\, P_{n-1,\lambda}(x)\, P_{n+1,\lambda}(x)$

Substituting for $P_{n-1,\lambda}$ and $P_{n+1,\lambda}$ from (5.1) we have

(5.5) $k_{n\lambda}\, \Delta_{n\lambda} = n\,(n+2\lambda)\, P^2_{n,\lambda} - \{nxP_{n,\lambda} + (1-x^2)\, P^1_{n,\lambda}\}\{n+2\lambda)\, x\, P_{n,\lambda} - (1-x^2)\ P'_{n,\lambda}\}$

$= (1-x^2)\left[n(n+2\lambda)P^2_{n,\lambda} + P^1_{n,\lambda}\ \{(1-x^2)P^1_{n,\lambda} - 2\lambda x P_{n,\lambda}\}\right]$

We now introduce the function

(5.6) $D_{n,\lambda}(x) \equiv P'^2_{n,\lambda}\ (x) - P'_{n-\lambda}\ (x)\, P'_{n+1,\lambda}\ (x)$

Substituting for $P'_{n-1,\lambda}$ and $P'_{n+1,\lambda}$ from the relations

(5.7) $x = P'_{n-1,\lambda} = x\ \ \ P'_{n,\lambda} - n\, P'_{n,\lambda}\ ,\ P'_{n+1,\lambda}, = x\, P'_{n,\lambda} + (n+2\lambda)\, P_{n\lambda}$,

We have

(5.8) $D_{n,\lambda} = P'_{n,\lambda}\ - \{xP'_{n,\lambda} - nP_{n,\lambda}\}\ \{(xP'_{n,\lambda} + (n+2\lambda)P_{n,\lambda}\}$

$= P'_{n,\lambda}\ \{(1-x^2)\, P'_{n,\lambda}\ - 2\lambda x P_{n,\lambda}\} + n\,(n+2\lambda)P^2_{n,\lambda}$

Comparison of (5.5) and (5.8) gives the relation

(5.9) $K_{n,\lambda}\, \Delta_{n,\lambda}\ (x) = (1-x^2)\, D_{n,\lambda}\ (x)$

Let n > 1. Taking derivatives in (5.8) and using (5.2), we have

(5.10) $D'_{n,\lambda} = P''_{n,\lambda}\{(1-x^2)P'_{n,\lambda} - 2\lambda x P_{n,\lambda}\} + P'_{n,\lambda}\{n(n+2\lambda)P_{n,\lambda} - xP'_{n,\lambda} - 2\lambda P'_{n,\lambda}\}$

$2\ \lambda\ 2\lambda\left[x(P'^2_{n,\lambda} - P_{n,\lambda}\, P''_{n,\lambda}) - P_{n,\lambda}\, P'_{n,\lambda}\right] =$

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Now we know that the roots of $P_{n,\lambda}(x)$ are all real and simple and lie in -1 < x < + 1 symmetrically w.r.t. the origin. Denoting them by $x_{\lambda n}^{n}$ (ν =1, …, n) we have

$$\frac{P'_{n,\lambda}}{P_{n,\lambda}}=\sum_{\gamma=1}^{n}\frac{1}{x-x_{n,\lambda}^{(\gamma)}},\quad \frac{P'^2_{n,\lambda}-P_{n,\lambda}P''_{n,\lambda}}{P^2_{n,\lambda}}=\sum_{\gamma=1}^{n}\frac{1}{(x-x_{n,\lambda}^{(\gamma)})^2},$$

Hence (5.10) may be written

$$D'_{n,\lambda}=2\lambda P^2_{n,\lambda}\left\{\sum_{\gamma-1}^{n}\frac{x}{(x-x_{n,\lambda}^{(\gamma)})^2}-\frac{1}{x-x_{n,\lambda}^{(\gamma)}}\right\}=2\lambda P^2_{n,\lambda}\sum_{\gamma-1}^{n}\frac{x_{n,\lambda}^{(\gamma)}}{(x-x_{n,\lambda}^{(\gamma)})^2}$$

We now take the roots $x_{n,\lambda}^{(\gamma)}$ in the form

$$\begin{cases} -\xi_{n,\lambda}^{(\mu)},\ +\xi_{n,\lambda}^{(\mu)}\ \ when\ \ n=2m \\ \qquad\qquad\qquad\qquad (\mu=1,\dots,m) \\ -\xi_{n,\lambda}^{(\mu)},\ 0+\xi_{n,\lambda}^{(\mu)}\ \ when\ \ n=2m+1 \end{cases}$$

Then, in either case, we have

$$(5.11)\quad D'_{n,\lambda}=2\lambda\sum_{\mu-1}^{m}\frac{\xi_{n,\lambda}^{(\mu)}}{(x-\xi_{n,\lambda}^{(\mu)})^2}-\frac{\xi_{n,\lambda}^{(\mu)}}{(x+\xi_{n,\lambda}^{(\mu)})^2}=8\lambda x\sum_{\mu=1}^{m}\xi_{n,\lambda}^{(\mu)^2}\ P_{n,\lambda}^{(\mu)^2}$$

Where the $P_{n,\lambda}^{(\mu)}$ are real polynomials defined by

$$P_{n,\lambda}(x) = (x^2 - \xi_{n,\lambda}^{(\mu)^2})\ P_{n,\lambda}^{(\mu)}(x)\ )$$

From (5.11), we see that

(5.12) sgn $D'_{n,\lambda}(x)$ = sgn (λx)

Now the following facts concerning the sign of $D_{n,\lambda}(x)$ at x = 0, ± 1, ± ∞ can be easily verified ($-\frac{1}{2}<\lambda\neq 0$) :

(5.13) $D_{n,\lambda}(0) > 0$, $D_{n,\lambda}(\pm\infty) = (\text{sgn}\,\lambda).\ \infty$, $D_{n,\lambda}(\pm 1) > 0$.

From (5.12) and (5.13), we arrive, by a consideration of the graph at the following results:

(5.14a) for $-\frac{-1}{2}<\lambda<0$, $D_{n,\lambda}(x)$ is increasing for x < 0, increasing for $x > 0$ and has its maximum for $x = 0$, resulting in the inequality : $D_{n,\lambda}(x) > 0$, $-1 \le x \le +1$

Which cannot be reversed for $|x| > 1$

(5.14b) for $\lambda = 0$, $D_{n,\lambda}(x)$ is decreasing for $x < 0$, increasing for $x > 0$ and has its minimum for $x = 0$ resulting in the – inequality $D_{n,\lambda}(x) > 0$ for all x.

With (5.14a, b), we get (1.2) along with the additional information that when n > 1 the inequality (1.2) is reversed for $|x| > 1$ if and only if $\lambda > 0$. Moreover, we get the following estimate for $\Delta_{n,\lambda}(x)$ in $-1 \le x \le +1$:

$$(5.15)\quad \begin{cases} \dfrac{1-x^2}{1+2\lambda}\le\Delta_{n,\lambda}(x)\le(1-x\gamma)\,\Delta_{n,\lambda}(0),-\dfrac{1}{2}<\lambda<0; \\ (1-x\gamma)\,\Delta_{n,\lambda}(0)\le\Delta_{n,\lambda}(x)\le\dfrac{1-x^2}{1+2\lambda},\ \gamma<0. \end{cases}$$

This follows immediately, in view of the relation (5.9), from the result obtained above that $D_{n,\lambda}(x)$ for $|x|\le 1$ is comprised between

$$D_{n,\lambda}(0) = k_{n,\lambda}\ \ \Delta_{n,\lambda}(0) \text{ and } \Delta_{n,\lambda}(1) = \frac{k_{n,\lambda}}{1+2\lambda}$$

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6. SOME IDENTITIES YIELDING PROOFS OF THE MAIN INEQUALITIES.

I. ULTRASPHERICAL POLYNOMIALS. The $F_{n,\lambda}(x)$ introduced in (1.2) satisfy the recursion :
(6.1) $(n + 2\lambda)\ F_{n+1,\lambda} - 2(n+\lambda)\ x\ F_{n,\lambda} + n\ F_{n-1,\lambda} = 0$
With $F_{o,\lambda} = 1$, $F_{1\lambda}, = x$. Changing n into n -1, we have
(6.2) $(n + 2\lambda - 1)\ F_{\lambda} - 2\ (n + \lambda - 1)\ n\ F_{n-1,\lambda} + (n-1)\ F_{n-2,\lambda} = 0,\ n \geq 1.$
With $F_{-1,\lambda}= 0$, $F_{o,\lambda} = 1$. Multiplying (6.2) by $F_{n,\lambda}$, (6.1)
By $F_{n-1,\lambda}$ and subtracting, we get the relation
(6.3) $(n+2\lambda)= \Delta_{n,\lambda} - (n-1)\ \Delta_{n-1,\lambda}= F_{n-1,\lambda}^2 - 2x\ F_{n-1},\ F_{n,\lambda} + F_{n,\lambda}^2$, $n \geq 1$

With $\Delta_{o,\lambda} = 1$. Since the right member there is evidently non-negative for $|x| \leq 1$, we get the inequality:
(6.4) $(n + 2\lambda)\ \Delta_{n,\lambda}\ (x) \geq (n-1)\ \Delta_{n-1,\lambda}\ (x),\ n \geq 1,\ |x| \leq 1.$

This immediately establishes (1.2) for $\lambda > -\dfrac{1}{2}$ by successive induction, and at the same time yields the more informative estimate:

(6.5) $\Delta_{n,\lambda}\ (x) \geq \dfrac{(n-1)!}{(1+2\lambda)...(n+2\lambda)}\ (1-x^2),\ n > 1,\ |x| \leq 1$

Again, changing n into n + 1 in (6.3), we have
(6.6) $(n + 2\lambda + 1)\ \Delta_{n+1,\lambda} - n\ \Delta_{n,\lambda} = F_{n+1,\lambda}^2 - 2x\ F_{n+1,\lambda}\ F_{n,\lambda} + F_{n,\lambda}^2$
If we now subtract (6.3) from (6.6) and use the relations (G. (S.1)]
(6.7) $(1-x^2)\ F_{n,\lambda}^{'} = n\ (F_{n-1} - x\ F_{n,\lambda}) = (n + 2\lambda)\ (xF_{n,\lambda} - F_{n+1,\lambda})$
We get

(6.8) $(n+2\lambda+1)\ \Delta_{n+1,\lambda}\ \ 2\ (n+\lambda)\ \Delta_{n,\lambda} + (n-1)\ \Delta_{n-2,\lambda}= -\dfrac{4\lambda(n+\lambda)}{n^2(n+2\lambda)^2}(1-x^2)^2\ F_{n,\lambda}^{;2}$ $(n \geq 1)$.

Hence the inequality:

(6.9) $(n+2\lambda+1)\ (\Delta_{n+1,\lambda} -\Delta_{n,\lambda}) \geq$ or $\leq (n-1)\ (\Delta_{n,\lambda} -\Delta_{n-1,\lambda})$, $n \geq 1$, according as $-\dfrac{1}{2} < \lambda < 0$ or $\lambda > 0$.

This gives us the additional information that for all x, $\Delta_{n,\lambda}(x)$ is a non decreasing function of n when λ 0.

Consider next the polynomials $P_{n-1,\lambda}^{'}(x)$. They satisfy the recursion:

(6.10) $n\ P_{n+1,\lambda}^{'} - 2(n+\lambda)x\ P_{n-1,\lambda}^{'} = 0$, $n \geq 1$, with $P_{0,\lambda}^{'} = 1$, $P_{1,\lambda}^{'} = 2\lambda x$. This follows from (5.7). From (6.10), we can easily derive the following relation, just as (6.3) was derived from (6.1) :

(6.11) $n\ D_{n,\lambda}^{'} - (n+2\lambda-1)\ D_{n-1,\lambda} = P_{n-1,\lambda}^{'2} - 2xP_{n-1,\lambda}^{'}\ P_{n,,\lambda}^{'} + P_{n,\lambda}^{'2}, n \geq 1,$

With $D_{0,\lambda} = 0$. We can then derive the following relation from (6.11) by using (5.7), just as (6.8) was derived from (6.3) by using (6.7) :
(6.12) $(n+1)\ D_{n+1,\lambda} -2(n+\lambda)D_{n,\lambda} + (n+2\lambda-1)\ D_{n-1,\lambda} = 4\lambda(n+\lambda)\,P_{n,,\lambda}^2$ $n \geq 0.$

With $D_{-1\lambda} = D_{0,\lambda} = 0$. From (6.12), it is now easy to deduce the following identity we have in view :

(6.13) $g_{n,\lambda}\ D_{n,\lambda}\ (x) = 4\lambda \sum_{\gamma=0}^{n-1} \left(\dfrac{g_{\gamma+1,\lambda}}{\gamma+1} + \dfrac{g_{\gamma+3,\lambda}}{\gamma+2} + ... + \dfrac{g_{n,\lambda}}{n} \right) (\gamma+\lambda) P_{r,\lambda}^2 | x),$

Where $g_{n,\lambda} = \dfrac{x!}{2\lambda(2\lambda+1)...(2\lambda+n-1)}$ $(n \geq 1)$.

Herewith, the positivity of $D_{n,\lambda}(x)$ for all x is rendered intuitive when $\lambda > 0$. Further more, in view of the relation (5.9), we recover the following complement to (1.2) for $\lambda > 0$.

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(6.14) $\Delta_{n,\lambda}(x) \gtreqless 0$ according as $|x| \gtreqless 1$.

In particular for $\lambda = \frac{1}{2}$ (the Legendre polynomial), we have

(6.15) $\Delta_n(x) = \frac{1-x^2}{n(n+1)} \sum_{\gamma=0}^{n-1} \left(\frac{1}{\gamma+1} + \frac{1}{\gamma+2} + \ldots + \frac{1}{n} \right) (2\gamma+1) P_{r,}^2(x)$,

We may also mention, without proof, the following identity – which serves the same purpose as (6.13) :

(6.16) $\frac{g_{n,\lambda}}{n+\lambda} D_{n,\lambda}(x) = \lambda \left[\sum_{\gamma=1}^{n-1} (\alpha_{r,\lambda} - \alpha_{r+1,\lambda}) P_{r,\lambda}^{'2}(x) + \alpha_{n,\lambda} P_{\gamma,\lambda}^{'2}(x) \right]$

With $\alpha_{n,\lambda} = \frac{g_{n,\lambda}}{n(n+\lambda-1)(n+\lambda)} (n \geq 1)$, $\overset{g_{n,\lambda}}{\wedge}$ having the same meaning as in (6.13).

LAGUERRE POLYNOMIALS: The analogue of Turán's – inequality (1.1) for the Laguerre polynomials $L_n^{(\alpha)}(x)$ reads:

(6.17) $\Delta_n^{(\alpha)}(x) \equiv \left\{ \Lambda_n^{(\alpha)}(x) \right\}^2 - \Lambda_{n-1}^{(\alpha)}(x) \Lambda_{n+1}^{(\alpha)}(x) \geq 0, \alpha > -1, n \geq 1$

Where $\Lambda_n^{(\alpha)}(x) \equiv L_n^{(\alpha)}(x) / L_n^{(\alpha)}(0)$. The $\Lambda_n^{(\alpha)}$ defined here satisfy the recursion:

(6.18) $(\alpha+n+1)\ \Lambda_{n+1}^{(\alpha)}$ - $(\alpha+2n+1-x)\ \Lambda_n^{(\alpha)}$ +n $\Lambda_{n-2}^{(\alpha)}$ = 0 n ≥ 1 with $\Lambda_0^{(\alpha)} = 1$, $\Lambda_1^{(\alpha)} = 1 - \frac{x}{\alpha+1}$ changing n into n-1, we have

(6.19) $(\alpha + n)\ \Lambda_n^{(\alpha)} - (\alpha + 2n - 1 - x) \Lambda_{n-1}^{(\alpha)} + (n-1) \Lambda_{n-1}^{(\alpha)} = o, n \geq 1$

with $\Lambda_{n-1}^{(\alpha)} = 1$, $\Lambda_1^{(\alpha)} = 0 = 1x\ \frac{x}{\alpha+1}$ Multiplying (6.19) by $\Lambda_n^{(\alpha)}$,

(6.19) by $\Lambda_{n-1}^{(\alpha)}$ and subtracting, we get the relation

(6.20) $(\alpha + nb + 1) \Delta_n^{(\alpha)} - (n-1)\ \Delta_{n-1}^{(\alpha)} = \left\{ \Delta_n^{(\alpha)} - \Delta_{n-1}^{(\alpha)} \right\}^2 n \geq 1$

With $\Delta_n^{(\alpha)} = 1$. This at once leads to the identity :

(6.21) $g_n^{(\alpha)}\ \Delta_n^{(\alpha)}(x) = \sum_{\gamma=1}^{n} \frac{g_v^{(\alpha)}}{\alpha+\gamma+1} \left\{ \Lambda_\gamma^{(\alpha)} - \Lambda_{\gamma-1}^{(\alpha)} \right\}^2$

Where $g_n^{(\alpha)} = \frac{(\alpha+1)\ldots(\alpha+n+1)}{(n-1)!}$ Herewith (6.17) is made evident. In particular for α = o (the ordinary Laguerre polynomial $L_n(x)$) we have

(6.22) $n(n+1)\ \left[L_n^2 - L_{n-1} L_{n+1} \right] = \sum_{\gamma=1}^{n} \gamma (L_\gamma - L_{\gamma-1})^2$

It also appears from that is true for -2 < α < - 1 is well.

III. BESSEL FUNCTIONS: It we put $\Delta_\gamma = J_\gamma^x - J_{\lambda-1} J_{\lambda+1}$ it can rewrite the relation (3.4) already noted for the Bessel function $J_\gamma(x)$ in the form

$(\gamma+1)\Delta_\gamma - \gamma\Delta_{\gamma+1} = J_\gamma^2 + J_{\gamma+1}^2$

This immediately leads to the relation

$$\frac{\Delta_\gamma}{\lambda} - \frac{\Delta_{\gamma+n}}{\lambda+n} = \sum_{\gamma-1}^{n} \frac{J_{\gamma+\gamma-1}^2 + J_{\gamma+\gamma}^2}{(\gamma+r-1)(\gamma+r)}$$

Since $J_{\gamma+n}(x) \to 0$ as $n \to \infty$, yields the following – series for Δ_γ -

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$$\Delta_\gamma = V \sum_{n-1}^{\infty} \frac{J^2_{\gamma+n-1} + J^2_{\gamma+n}}{(\gamma+n-1)(\gamma+n)}$$

What is easily seen to be equivalent to

$$J^2_\gamma - J_{\gamma-1} J_{\gamma+1} = \frac{1}{\gamma+1} J^2_\gamma + \frac{2}{\gamma+2} J^2_{\gamma+1} + 2\gamma \sum_{n-2}^{\infty} \frac{J^2_{\gamma+n}}{(\gamma+n-1)(\gamma+n+1)} \quad J_{\gamma-1}\ \text{o}$$

This series representation of Δ_γ is valid unless γ is a negative integer, in which cane it holds with - γ in place of γ since Δ_γ is an even function of γ for integral values of γ_1.

For $\gamma \geq 0$, at once disposes of Szász's inequality (1.6). It is now all the more significant to note that our inequality (1.8), which asserts the positivity of Δ_γ even for $-1 < \gamma < 0$, is not at all placed in evidence by like Szász's inequality.

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