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The Mobius Strip's Shape

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ABSTRACT
The Möbius strip, obtained by taking a rectangular strip of plastic or paper, twisting one end through 180°, and then joining the ends, is the canonical example of a one-sided surface. Finding its characteristic developable shape has been an open problem ever since its first formulation. Here we use the invariant variational bicomplex formalism to derive the first equilibrium equations for a wide developable strip undergoing large deformations, thereby giving the first non-trivial demonstration of the potential of this approach. We then formulate the boundary-value problem for the Möbius strip and solve it numerically. Solutions for increasing width show the formation of creases bounding nearly flat triangular regions, a feature also familiar from fabric draping and paper crumpling. This could give new insight into energy localization phenomena in unstretchable sheets, which might help to predict points of onset of tearing. It could also aid our understanding of the relationship between geometry and physical properties of nano- and microscopic Möbius strip structures

1.Introduction
The Mobius strip is one of the few mathematical symbols that has successfully entered popular culture. Its mathematical elegance has influenced artists like M.C. Escher. In order to wear "both" sides equally, Mobius strips are frequently used as pulley belts in engineering. Mobius strips have recently formed at a much smaller scale in ribbon-shaped NbSe3 crystals under specific growth conditions involving a significant temperature gradient. Tanda et al. suggest a combination of Se surface tension, which bends the crystal, and twisting as a result of bend-twist coupling because the ribbon is a crystal, as the explanation for this behaviour. The quantum eigenstates of a particle contained on the surface of a developable Mobius strip were recently calculated by Gravesen & Willatzen, and the results were contrasted with earlier calculations by Yakubo et al. The groundstate wavefunction, which was otherwise doubly degenerate, was split, indicating the presence of curvature effects.

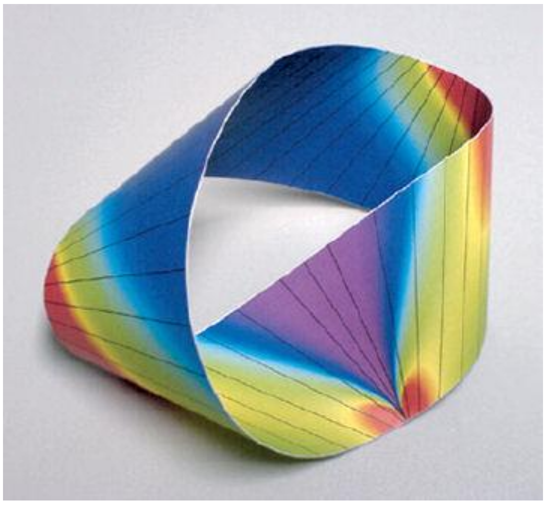

Fig 1. A 2π Mobius Strip made with paper

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2. Geometrical Model of Mobius Strip

The most basic geometric representation of a Mobius strip is a ruled surface that is swept out by a normal vector travelling along a closed path while making a half-turn. This model does not adequately describe a typical paper Mobius strip because the surface it generates need not be developable, which prevents it from being mapped isometrically (i.e., with preservation of all intrinsic distances) to a plane strip. Because it is more efficient to bend a piece of paper than to stretch it, a paper strip can be created roughly. Therefore, the strip deforms in a way that barely affects its metrical characteristics. There is a good chance that some nano structures share the same elastic characteristics. The disappearance of the surface's Gaussian curvature everywhere is a prerequisite for a surface to be developable. There is a specific flat ruled surface (referred to as the "rectifying developable") on which a geodesic curve exists given a curve with non-vanishing curvature. Examples of analytic (and even algebraic) developable Mobius strips have been created using this property.

3. Equations used for Mobius Strip

If $r(s)$ is a parametrisation of a curve then $x(s, t) = r(s) + t[b(s) + \eta(s)\, t(s)]$, $\tau(s) = \eta(s)\kappa(s)$, $s = [0, L]$, $t = [-w, w]$ (1)

is a parametrisation of a strip with r as centreline and of length L and width 2w, where t is the unit tangent

vector, b the unit binormal, κ the curvature and τ the torsion of the centreline . The parametrised lines s = const. are the generators, which make an angle $\beta = \arctan(1/\eta)$ with the positive tangent direction. Thus, the shape of a developable Mobius strip is completely determined by its centreline. We also recall that a regular curve in 3D is completely determined (up to Euclidean motions) by its curvature and torsion as functions of arclength. An actual material Mobius strip made of inextensible material, as demonstrated by straightforward experimentation, takes on a distinctive shape when left to itself, regardless of the type of material (enough stiff for gravity to be ignorable). The deformation energy, which is entirely a result of bending, is minimised by this shape. We'll assume that the material will bend in accordance with Hooke's linear law. The elastic energy is then proportional to the integral of the other principal curvature squared over the surface of the strip because one of the principal curvatures for a developable surface is zero.

$V = 1/2\, D \int_0^L \int_{-w}^{w} \kappa_1^2(s, t)\, dt ds$ where $D = 2h\, 3E/[3(1-\nu\, 2\,)]$, with 2h the thickness of the strip, and E and ν Young's modulus and Poisson's ratio of the material.

3.1 Energy Minimisation

Energy minimisation is thus turned into a 1D variational problem represented in a form that is invariant under Euclidean motions. Even with the aid of contemporary symbolic computer software, the standard method of solving it expressing the Lagrangian g in terms of r and its derivatives (or perhaps introducing coordinates) and deriving the Euler-Lagrange equations is challenging, and there don't appear to be any equations for the finitewidth case in the literature. Here, we employ a potent geometric strategy based on the variational bicomplex formalism, enabling us to quickly obtain a manageable set of equations in invariant form. When applied to variational problems for space curves, this theory apparently little-known outside of the mathematic community produces equilibrium equations for functionals of a general type.

$\int_0^L f(\kappa, \tau, \kappa', \tau', \kappa'', \tau'', \cdots, \kappa(n), \tau(n))\, ds$

involving derivatives up to any order . While a similar method was used to derive Euler-Lagrange equations for some straightforward Lagrangians f, our current issue seems to be the first where an invariant approach is absolutely necessary to find a solution. Randrup and Rogen have demonstrated that an odd number of switching points along the centerline of a rectifying developable Mobius strip must take place where = = 0 and the principal normal to the centerline flips (i.e., 180 degrees). As a result, the strip needs to have an umbilic line, or a point where both of the main curves disappear. (Incidentally, a Mobius strip may be built that has no switching points if the initial strip is not a

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rectangle.) We note that a closed centerline with a periodic twist rate (here, (s)) defines a closed cord [9], for which one can define a linking number Lk in order to precisely describe the twisted nature of the Mobius strip. Any half-integer Lk cord or ribbon has one side only.

3.2 3D shapes of Mobius Strip

The centreline in 3D may be reconstructed from (κ(s), τ(s)) by integrating the usual Frenet-Serret equations and the equation r ′ = t. Combining these results into a differential-algebraic system of equations allows us to formulate a boundary-value problem for the Mobius strip, for which we impose boundary conditions at s = 0 and s = L/2 and choose the solution with Lk = 1 /2. The solution is then obtained on the entire [0, L] interval by appropriate reflection using the involution property. This results in a symmetric solution; non-symmetric solutions seem unlikely to exist.

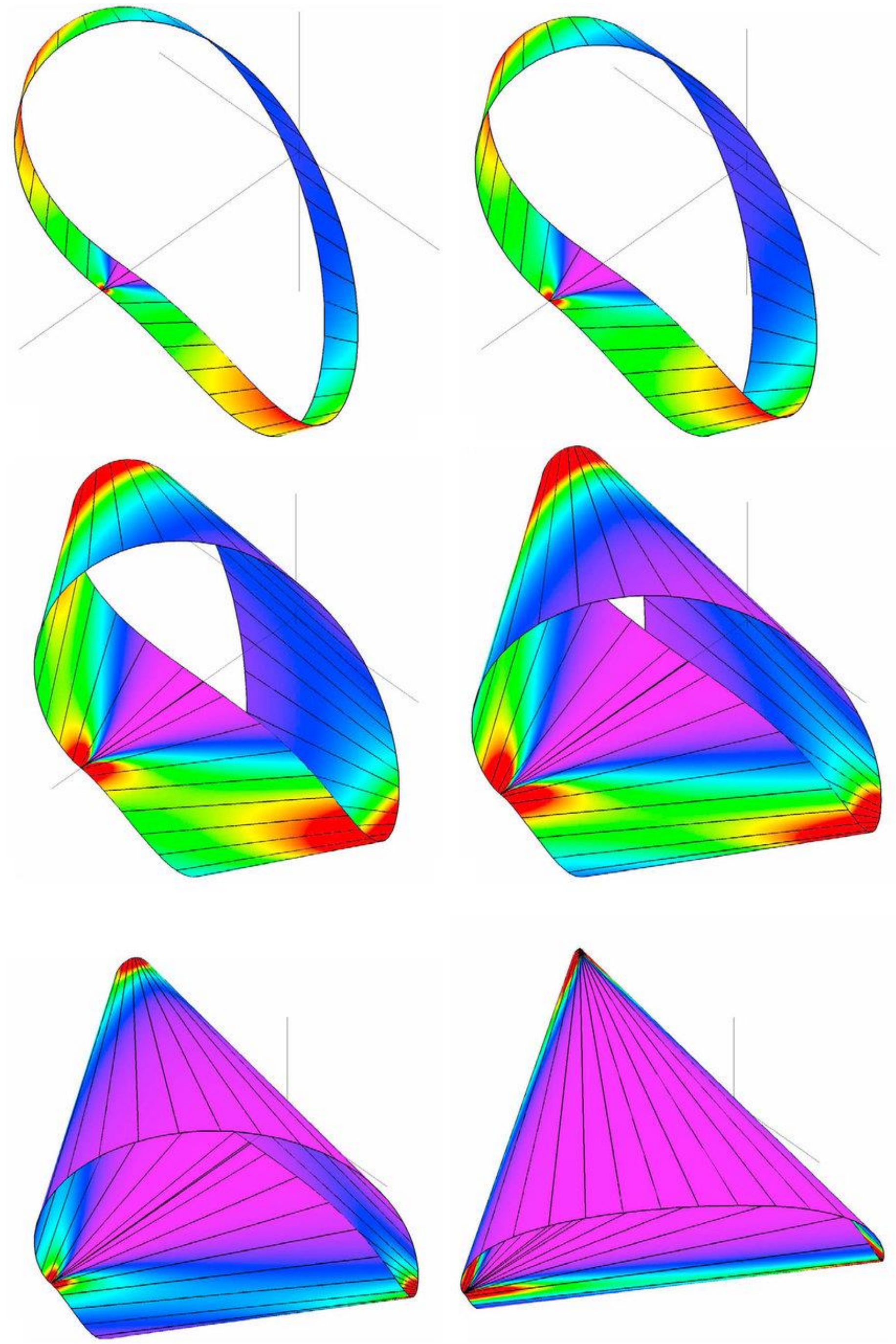

Figure 2. Different 3D shapes for various *w* values

A numerically obtained solution is shown in Fig. 2. The aspect ratio L/2w of the strip is the only physical variable in the issue. We varied w and fixed L = 2 in the calculations. The evolution along the strip of the straight generator is also depicted in the figures. We mark the places where the generators begin to build up. At these locations, the integrand in (3) (the energy density) diverges and $|w\eta'| \rightarrow 1$. The generator quickly sweeps through a nearly flat (violet) triangular region where this occurs, a phenomenon that can be easily seen in a paper Mobius strip (Fig. 1). Additionally, we see two (milder) accumulations where there is no inflection and the energy density is constant. The monotonicity of the energy density along a generator can be demonstrated. This suggests that a generator cannot connect the (red) regions of high curvature, as a close look demonstrates. Two generators of constant curvature surround the (violet) triangular (more precisely, trapezoidal) regions. Local minima for the angle β are realised by these generators.
The accumulations and related triangular regions become more obvious as w is increased. The strip collapses into an equilateral triangle with triple coverage at the critical value indicated by w/L =3/6. The tightening of tubular knots as they get closer to the ideal shape of a minimum length to diameter ratio can be compared to the folding process as w is increased towards this flat triangular limit. The generators are split into three groups and intersect each other at three vertices in the flat limit. The creases are formed by the constant curvature bounding generators.

Conclusion

The geometrical characteristics of Mobius strips are more commonly observed in elastic sheet problems like paper folding or crumpling and fabric draping. We can observe this behaviour in the nearly flat triangular regions of Figure 2 because paper crumpling is primarily caused by bending along ridges enclosing almost flat regions or facets. In the process of fabric draping, triangular regions are observed to emerge from the approximate vertices. These flat triangular regions are thought to be the result of nature's reaction to the twisting of inextensible sheets. Analysis of these sheets frequently makes use of conical surface vertices as regions of localised bending energy**.**
Within the framework of the linear elastic theory, conical surfaces are known to possess infinite elastic energy. The introduction of a cut-off is necessary due to the difficulties this causes. As demonstrated by the Mobius strip example, one can describe bending localization phenomena without a cut-off by taking into account non-conical developable elastic surfaces. Importantly, our method foresees the emergence of high bending regions. Points of divergence in the bending energy could be used to identify potential locations for the onset of fracture failure mode III, or out-of-plane tearing. It's interesting to note in this regard that when attempting to tear a piece of paper(fig 3), one naturally applies a torsion, resulting in intersecting creases, as seen in the vertices of the central triangular domains in Fig. 2.

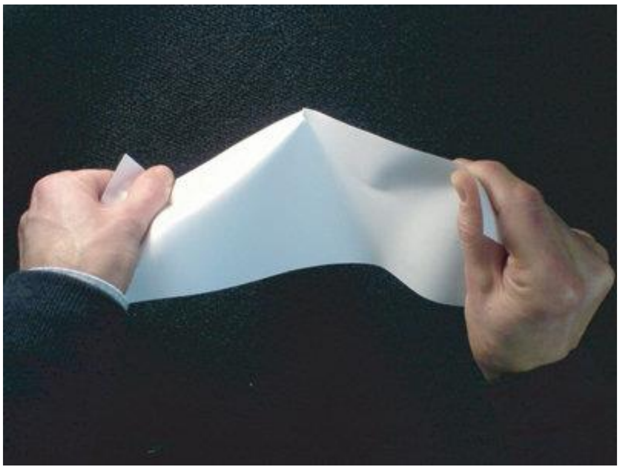

Fig 3. While trying to tear a piece of paper we can see a shape similar to the mobius strip is formed.

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