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A Comparative Study of Proofs on Liouville's Theorem

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ABSTRACT
We will have a study on the different proofs of Liouville's theorem. There are many different ways to prove Liouville's theorem. Here we study proofs using Cauchy's integral formula, Cauchy's estimate and by using Picard's Little theorem. While comparing these proofs we can conclude that if we use Picard's Little theorem, we can prove Liouville's theorem more easily.
Keywords—Liouville's theorem, Picard's Theorem

1. Introduction

Liouville's theorem is given by Joseph Liouville. He was a French mathematician and Engineer. Liouville's theorem was presented by Liouville in his lectures in 1847, although it was believed to be first proved by famous mathematician A.L. Cauchy in 1844.

Liouville's theorem is concerned with the entire function being bounded over a given domain in a Complex plane. An entire function is a complex analytic function that is analytic throughout the whole Complex plane. For example, exponential function, *sin z, cos z* and polynomial functions.

2. Experimental Methods or Methodology

Liouville's Theorem

A function which is analytic and bounded in the whole complex plane must reduce to a constant.[1]

Different proofs of Liouville's theorem

a) Proof using Cauchy's integral formula:

Given f is an analytic function in the whole complex plane.

i.e., f is an entire function.

Also given that f is a bounded function, i.e. $f(z) \leq k$, $\forall\ z \epsilon C$

By Cauchy's integral formula $f(a)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz$

$$f(b)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}dz$$

$$f(a)-f(b)=\frac{1}{2\pi i}\int_C \frac{f(z)}{z-a}dz - \frac{1}{2\pi i}\int_C \frac{f(z)}{z-b}dz$$

$$=\frac{1}{2\pi i}\int_C \left(\frac{f(z)}{z-a}-\frac{f(z)}{z-b}\right)dz$$

$$=\frac{1}{2\pi i}\int_C f(z)\left(\frac{1}{z-a}-\frac{1}{z-b}\right)dz$$

$$=\frac{1}{2\pi i}\int_C f(z)\left(\frac{(z-b)-(z-a)}{(z-a)(z-b)}\right)dz$$

$$=\frac{1}{2\pi i}\int_C f(z)\left(\frac{a-b}{(z-a)(z-b)}\right)dz$$

$$=\frac{1}{2\pi i}\int_C \frac{f(z).(a-b)}{(z-a)(z-b)}dz$$

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$f(a)-f(b) = \frac{(a-b)}{2\pi i}\int_C \frac{f(z)}{(z-a)(z-b)}dz$ Circle: |z|=R

$|f(a)-f(b)| = |\frac{(a-b)}{2\pi i}\int_C \frac{f(z)}{(z-a)(z-b)}dz|$ $z=Re^{i\theta}$, $dz=i\,Re^{i\theta}d\theta, |dz| = Rd\theta$

$\leq \frac{|a-b|}{|2\pi i|}\int_C \frac{|f(z)|}{|z-a||z-b|}|dz|$

$\leq \frac{|a-b|}{2\pi}\int_C \frac{k}{(|z|-|a|)(|z|-|b|)}.R\,d\theta$

$\leq \frac{|a-b|}{2\pi}\int_C \frac{k}{(R-|a|)(R-|b|)}.R\,d\theta$

$\leq \frac{k\,|a-b|}{2\pi}\int_C \frac{R\,d\theta}{(R-|a|)(R-|b|)}$

$\leq \frac{k\,|a-b|}{2\pi}\lim_{R\to\infty}\int_C \frac{R\,d\theta}{(R-|a|)(R-|b|}$

$\leq \frac{k\,|a-b|}{2\pi}\int_C \lim_{R\to\infty}\frac{R\,d\theta}{(R-|a|)(R-|b|)}$

$\leq \frac{k\,|a-b|}{2\pi}\int_C \lim_{R\to\infty}\frac{R\,d\theta}{R(1-\frac{|a|}{R})R(1-\frac{|b|}{R})}$

$|f(a)-f(b)| \leq \frac{k\,|a-b|}{2\pi}\int_C \lim_{R\to\infty}\frac{d\theta}{R(1-\frac{|a|}{R})(1-\frac{|b|}{R})}$

≤ 0

$\therefore\ f(a)-f(b)=0$

$f(a)=f(b)$

Since a and b are arbitrary complex numbers and *f(a)=f(b)*

$$\therefore f(a) = f(b), \forall a,b \ \in\in C$$

Hence we can conclude that *f* is a constant.

Cauchy's estimate[3]

Suppose *f* is analytic within and on $C = \{z: |z - a| = R\}$ and suppose that

Then $|f^{(n)}(a)| \leq \frac{n!M_R}{R^n}$,n=0,1,2,…..

Where $M_R = \max_{z\in C}|f(z)|$

b) Proof using Cauchy's estimate

Suppose $f : C \to C$ is everywhere differentiable and is bounded above by *M,* i.e. $|f(z)| \leq M$ for every $z \in C$. Fix an arbitrary a. Since *f* is holomorphic everywhere, it is in particular holomorphic on a neighbourhood of B (a, R) for any value of $R > 0$. By the Cauchy Estimates, since

$M_R = \{|f(z)|: |z-a|=R\} \leq M$

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We have, $|f^{1}(a)| \leq \frac{M_R}{R} \leq \frac{M}{R}$, $\forall R > 0$

Since the expression on the left is a nonnegative constant, letting $R \to \infty$ on the right yields

$$|f^{1}(a)| \leq 0$$

Hence $f^{1}(a) = 0$

But a was arbitrary, so $f^{1}(z) \equiv 0$ on C. Then $f(z)$ is necessarily a constant.

Picard's Little Theorem [5]

"A non-constant entire function takes on every value except at most one."

i.e., *f* an entire function omits more than one point in the complex plane, then it must be a constant.

Examples

Sin z is unbounded; *Cos z* is unbounded.

c) Proof Of Liouville's Theorem Using Picard's Little Theorem

Assume *f* is analytic and bounded in the whole Complex plane. We need to prove that *f* is a constant. Since *f* is bounded, *f* omits more than one value in *C*. Hence by Picard's Little Theorem, *f* must be a constant.

3. Results and Discussion

3.1 Cauchy's integral formula

We can use Cauchy's integral formula to prove Liouville's theorem. In order to prove Liouville's theorem using Cauchy's integral formula, first we assume that the function *f* is analytic and bounded and then by applying Cauchy's integral formula we will get that *f(a) = f(b)*, for any arbitrary a and b. That means *f* is a constant. But this method involves lot mathematical steps and formulas.

3.2 Cauchy's estimate

We can use Cauchy's estimate to prove Liouville's theorem. Here in order to prove Liouville's theorem using Cauchy's estimate, we will assume that *f* is analytic and bounded, by applying Cauchy's estimate we get $f^{1}(a)=0$, for arbitrary a. Since derivative of *f* is zero for every complex number, we can conclude that f is a constant. By using Cauchy's estimate we can prove Liouville's theorem in few mathematical computations compared to the proof using Cauchy's integral formula.

3.3 Picard's Little theorem

By using Picard's little theorem, we can directly prove the Liouville's theorem. As Picard's theorem assumes that if an analytic function omits more than one point then *f* must be a constant. Liouville's

4. Conclusion

As per preceding research and findings, we have observed that we can prove Liouville's theorem very easily using Picard's Little theorem.

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